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ON A PROBLEM OF OPTIMUM PRIORITY CLASSIFICATION

by

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ON A PROBLEM OF OPTIMUM PRIORITY CLASSIFICATION

1. Introduction and Summary

In certain traffic and storage operations many types of customers use a common service facility. At an airport runway, for example, landings and departures may consist of many types and sizes of propeller and jet aircraft, each with different service characteristics. It is often possible to assign each customer to a priority class $n = 1, 2, \ldots, N + 1$ and devise an ordered servicing rule, as a function of n, which leads to better performance of the service system than could be expected if customers were serviced in the order of their arrival.

When arrivals to the service facility are Poisson, service times of each priority class are independently distributed positive random variables, the order of service within each priority class is first-come, first-serve and the service of a customer, once begun, is never interrupted, expressions have been obtained by Cobham [1] and by Kesten and Runnenburg [3] for the stationary probability distribution and moments of waiting times of each priority class.

Under these same conditions Cox and Smith[2] have established that the priority rule which minimizes expected queueing time of all customers gives high priority to those classes with low mean servicing time. In other words, the optimum priority rule is simply one of ranking the mean service time of each priority class and does not require any restrictions, other than existence, on the distribution function or higher moments of the service time. In the proof that such a rule is

optimal it is assumed that the a priori arrival and service distributions of each class are known and are unchanging with time.

In some traffic systems, on the other hand, the arrival of a customer may lead to different a posteriori statements about his service time distribution. This new knowledge suggests that the overall performance of the service system may be improved if a customer, upon arrival, is segregated into a priority class.

In this paper we consider the extreme case where the service time of an arrival is known exactly once he joins the queue of the service facility; prior to this moment his service time is sampled from a stationary distribution function common to all customers. We pose the following problem: with a fixed number of priority classes, how should one assign priorities to customers in order to minimize expected queueing time of all customers using the service system.

It is shown that this decision problem can be formulated in terms of a non-preemptive priority queueing model and that the mathematical optimization can be expressed as a functional equation involving two variables: the number of priority classes and the truncation point which separates two priority classes. Using results from the theory of Dynamic Programming [5], it is possible to express results for the N priority class problem in terms of a two-class problem and obtain monotone sequences for average queueing time and the truncation point for each priority class. These results converge, in the limit of large N, to the results obtained earlier by Phipps[4].

2. Expected Queueing Time

The expected queueing time of customers in the jth of $N + 1$

priority classes in a Poisson-fed queue with non-preemptive service, each priority class having independently distributed service times with distribution function $B_j(x)$ is

(1a) $$w_j = \frac{\lambda \nu^{(2)}}{2(1-\gamma_{j-1})(1-\gamma_j)} \qquad j = 1, 2, \ldots, N+1$$

where $\gamma_j < 1$ and

(1b) $$\lambda = \sum_{j=1}^{N+1} \lambda_j = \text{total arrival rate into the system.}$$

(1c) $$\alpha_j = \lambda_j \lambda^{-1} = \text{the fractional arrival rate of the } j\underline{th} \text{ priority class.}$$

(1d) $$\frac{1}{\mu_j} = \text{average service time of the } j\underline{th} \text{ class.}$$

(1e) $$\nu_j^{(2)} = \text{second moment of the service time, } j\underline{th} \text{ class.}$$

(1f) $$\rho_j = \lambda_j/\mu_j \; ; \; \gamma_j = \sum_{i=1}^{j} \rho_i \; ; \; \gamma_o = 0 \, .$$

(1g) $$\frac{1}{\mu} = \sum_j \frac{\alpha_j}{\mu_j} \; ; \; \nu^{(2)} = \sum_j \alpha_j \nu_j^{(2)}$$

The expected queueing time for the service system is defined as the sum of the w_j's weighted by the fraction of arrivals into the $j\underline{th}$ priority class:

(2) $$W_{N+1} = \sum_{j=1}^{N+1} \alpha_j w_j = \frac{\lambda \nu^{(2)}}{2} \left\{ \sum_{j=1}^{N+1} \frac{\alpha_j}{(1-\gamma_{j-1})(1-\gamma_j)} \right\}$$

If we segregate the arrivals in a single stream of Poisson traffic

into N classes so that the customers in the nth priority class have service times in the interval (y_{n-1}, y_n) they constitute a fraction,

(3a) $$\alpha_n = \alpha(y_{n-1}, y_n) = B(y_n) - B(y_{n-1})$$

of all arrivals. The distribution of customer service times in this priority class is

(3b) $$B_n(x) = \begin{cases} 0 & 0 \leqq x \leqq y_{n-1} \\ \dfrac{B(x) - B(y_{n-1})}{B(y_n) - B(y_{n-1})} & y_{n-1} < x \leqq y_n \\ 1 & y_n < x \end{cases}$$

and hence the fractional utilization (1f) of the service system by the nth priority class is

(4a) $$\rho_n = \rho(y_{n-1}, y_n) = \lambda \int_{y_{n-1}}^{y_n} x \, dB(x)$$

(4b) $$\gamma_n = \gamma(y_n) = \lambda \int_0^{y_n} x \, dB(x) .$$

We see that the total arrival rate, λ, and the moments, $\frac{1}{\mu}$ and $\nu^{(2)}$ are independent of the choice of y_n. Substituting Equation (3a) and (4b) into (2) expresses W_{N+1} as a function of the n truncation points y_i and $y_{N+1} = \infty$.

(5) $$W_{N+1} = W_{N+1}(y_1, y_2, \ldots, y_N) = \frac{\lambda\nu^{(2)}}{2} \left\{ \sum_{n=1}^{N+1} \frac{\alpha(y_{n-1}, y_n)}{(1-\gamma(y_{n-1}))(1-\gamma(y_n))} \right\}$$

In the proofs which follow we assume continuity in the distribution

function $B(x)$ which implies continuity in $\alpha(y, x)$, $\gamma(x)$ and W_N .

3. A Functional Equation for Expected Queueing Times

The problem posed in the introduction to this paper is one of finding the location of the y_i (when they exist) which minimize expected queueing time in Equation (5).

$$W^*_{N+1} = \min_{0 \le y_1 \le \dots y_N \le \infty} W_{N+1} \tag{6}$$

By considering a service system with one less priority class, it is possible to imbed the solution of W^*_N in the solution for W^*_{N+1} . To simplify notation, let

$$f_{N+1}(x) = \min_{0 \leqq y_1 \leqq, \dots, y_N \leqq x} \sum_{n=1}^{N+1} \frac{\alpha(y_{n-1}, y_n)}{(1-\gamma(y_{n-1}))(1-\gamma(y_n))} \tag{7}$$

with the understanding that $\gamma_0 = 0$; $y_{N+1} = x$ and $\gamma(x) < 1$. $f_{N+1}(x)$ differs from W^*_{N+1} by the factor $\frac{\lambda\nu^{(2)}}{2}$ and the fact that the truncation point is $x < \infty$. To show that $f_{N+1}(x)$, and hence W_{N+1} satisfies

$$f_{N+1}(x) = \min_{0 \le y \le x} \left\{ f_N(y) + \frac{\alpha(y, x)}{(1-\gamma(y))(1-\gamma(x))} \right\} \tag{8}$$

we pick a sequence

$$0 \le x^*_1(x) \le x^*_2(x) \le \dots x^*_N(x) \le x$$

such that for any

$$0 \leqq x_1 \leqq x_2 \leqq \dots \leqq x_N \leqq x$$

$$(9) \qquad f_{N+1}(x) = \sum_{n=1}^{N+1} \frac{\alpha(x^*_{n-1}, x^*_n)}{(1-\gamma(x^*_{n-1}))(1-\gamma(x^*_n))} \leqq \sum_{n=1}^{N+1} \frac{\alpha(x_{n-1}, x_n)}{(1-\gamma(x_{n-1}))(1-\gamma_n(x_n))} .$$

Pick any number y and sequence

$$0 \leqq x'_1(y) \leqq x'_2(y) \ldots x'_{N-1}(y) \leqq y$$

such that for any

$$0 \leqq x_1 \leqq x_2 \leqq \cdots \leqq x_{N-1} \leqq y$$

$$f_N(y) = \sum_{n=1}^{N} \frac{\alpha(x'_{n-1}, x'_n)}{(1-\gamma(x'_{n-1}))(1-\gamma(x'_n))} \leqq \sum_{n=1}^{N} \frac{\alpha(x_{n-1}), x_n)}{(1-\gamma(x_{n-1}))(1-\gamma(x_n))}$$

Now, pick $0 \leqq y^* \leqq x$ such that

$$(10) \qquad f_N(y^*) + \frac{\alpha(y^*, x)}{(1-\gamma(y^*))(1-\gamma(x))} \leq f_N(y) + \frac{\alpha(y, x)}{(1-\gamma(y))(1-\gamma(x))} .$$

Since the inequality also holds when x^*_N is substituted for y in the right-hand side of (10) we have

$$(11) \qquad f_N(y^*) + \frac{\alpha(y^*, x)}{(1-\gamma(y^*))(1-\gamma(x))} \leq f_{N+1}(x)$$

But the defining equation (7) for $f_{N+1}(x)$ reverses the inequality of (11) and we have therefore shown that $f_{N+1}(x)$ is the solution of Equation (8):

$$f_{N+1}(x) = \min_{0 \leq y \leq x} \left\{ f_N(y) + \frac{\alpha(y, x)}{(1-\gamma(y))(1-\gamma(x))} \right\}$$

4. Results for the 2 Class Case

The structure of $f_{N+1}(x)$ and the location of the minimizing sequence $(y_1^*, y_2^*, \ldots, y_N^*)$ proceeds by induction on n from results for the two class case. We have

$$(12) \qquad f_2(x) = \underset{0 \leqq y \leqq x}{\mathrm{Min}} \left\{ f_1(y) + \frac{\alpha(y, x)}{(1-\gamma(y))(1-\gamma(x))} \right\}$$

$$= \underset{0 \leqq y \leqq x}{\mathrm{Min}} \left\{ \frac{\alpha(o,y)}{1-\gamma(y)} + \frac{\alpha(y, x)}{(1-\gamma(y))(1-\gamma(x))} \right\}$$

$$= \underset{0 \leqq y \leqq x}{\mathrm{Min}} \left\{ g_1(y,x) \right\} = g_1(y_1^*, x)$$

Assuming a continuous distribution function, an interior solution for y_1^* occurs when it is the solution of

$$(13) \qquad \frac{\gamma(x)}{B(x)} = \frac{\lambda y}{1 + \lambda y\, B(y) - \gamma(y)}$$

This result can be obtained by observing that the derivative of $g_1(y, x)$ with respect to y can be written as the product of two factors, one of which is nonnegative, the zeroes of the other corresponding to the relative extrema of $g_1(y, x)$. Substituting this implicit solution for y_1^* into Equation (12) gives

$$(14) \qquad f_2(x) = \frac{B(x)-\gamma(x)B(y_1^*)}{(1-\gamma(x))(1-\gamma(y_1^*))}$$

$$= \frac{1}{\lambda y_1^*} \cdot \frac{\gamma(x)}{1-\gamma(x)}$$

The functions $f_1(x)$, $f_2(x)$ are non-decreasing functions of x. Since $\alpha(0,x)$ is non-decreasing and $1 - \gamma(x)$ is non-increasing,

$f_1(x)$ is non-decreasing and its derivative can be written

$$\frac{df_1}{dx} = \frac{b(x)}{(1-\gamma(x))^2}\, u_1(x) \tag{15}$$

where $u_1(x) = 1 + \lambda \int_0^x B(t)dt$ is a convex function of x. The derivative of (14) for the two class case can also be expressed in terms of the optimal truncation point $y_1^*(x)$ to give

$$\frac{df_2}{dx} = \frac{\partial}{\partial x}\left\{ \frac{\alpha(y_1^*, x)}{(1-\gamma(y_1^*))(1-\gamma(x))} \right\} = \frac{b(x)}{(1-\gamma(x))^2}\, u_2(x) \tag{16a}$$

where

$$u_2(x) = \frac{1-\gamma(x) + \lambda x[B(x) - B(y_1^*)]}{1-\gamma(y_1^*)} \tag{16b}$$

As one might expect, $u_1(x)$ can be viewed as a special case of $u_2(x)$ where $y_1^*(x) = 0$.

To obtain upper and lower bounds on $u_2(x)$ we first show that $u_2(x) \geq 1$. Since the numerator of (16b) can be written

$$1-\gamma(x) + \lambda x[B(x) - B(y_1^*)] = 1-\gamma(y_1^*) + \lambda \int_{y_1^*}^x (x-t)\, dB(t)$$

it is obviously greater than or equal to $1-\gamma(y_1^*)$ for $y_1^* \leq x$, and thereby establishes a lower bound of unity for $u_2(x)$. We also note that $u_2(0) = u_1(0) = 1$.

To show that $u_2(x) \leq u_1(x)$ we assume the contrary; namely, that:

$$u_2(x) = \frac{u_1(x) - \lambda x B(y_1^*)}{1-\gamma(y_1^*)} > u_1(x) \qquad y_1^* \leq x \tag{17a}$$

The right hand inequality implies

(17b) $\lambda x\, B(y_1^*) < u_1(x)\, \gamma(y_1^*)$

which can be rewritten in the form

(17c) $x < u_1(x)\, \nu(y_1^*)$

with the understanding that $\nu(z) = B^{-1}(z) \int_0^z t\, dB(t)$ is the conditional mean service time of those customers having service times in $(0, z)$. The solution for y_1^* in Equation (13) can now be written as

(18) $y_1^* = u_1(y_1^*)\, \nu(x)$

and this equation in combination with (17c) and the fact that $\frac{u_1(y)}{y}$ is a decreasing function of y leads to the inequality

(19a) $\frac{1}{\nu(x)} = \frac{u_1(y_1^*)}{y_1^*} \geq \frac{u_1(x)}{x} > \frac{1}{\nu(y_1^*)} \qquad y_1^* \leq x$.

We observe, however, that the requirement that the two class system in (14) have average delay less than or equal to the one-class system provides the additional inequality.

(19b) $\frac{1}{\lambda y_1^*}\, \frac{\gamma(x)}{1-\gamma(x)} \leq \frac{B(x)}{1-\gamma(x)}$

which, by cancellation of the common term $1 - \gamma(x)$, implies that $y_1^* \geq \nu(x)$; i.e., that the point separating two priority classes lies to the right of the conditional mean. With the non-decreasing property of the conditional mean we therefore obtain the ranking

(19c) $\nu(y_1^*) \leq \nu(x) \leq y_1^* \leq x$.

The left-hand inequality of (19c) contradicts the strict inequality of the left and right hand side of (19a); hence the assumption that $u_2(x) > u_1(x)$ is false. To summarize the results of this section, we have obtained bounds on the truncation point $y_1^* = y_1^*(x)$ and shown that $1 \leq u_2(x) \leq u_1(x)$. The latter inequality implies that

(19d) $$\frac{df_1}{dx} \geq \frac{df_2}{dx} \quad \text{and} \quad f_1(x) \geq f_2(x) \ .$$

In the following section we obtain similar results for the model of $N > 2$ priority classes.

5. The N-Class Case

Denote by $y_N^* = y_N^*(x)$ the largest value of y which is a solution of

$$(20) \qquad f_{N+1}(x) = \underset{0 \leq y \leq x}{\text{Min}} \left\{ f_N(y) + \frac{\alpha(y\,,\,x)}{(1-\gamma(y))(1-\gamma(x))} \right\}$$

In other words y_N^* is the optimal truncation point separating the Nth and (N+1)st priority class where $y_N^* = \underset{i}{\text{Max}}\,(y^{(i)}(x))$ and $y^{(i)}(x)$ are solutions of (20). Generalizing our earlier notation we write

$$f_{N+1}(x) = \underset{0 \leq y \leq x}{\text{Min}} \; g_N(y\,,\,x)$$

and can only obtain an interior solution for y_N^* when

$$(21) \qquad \frac{df_N}{dy} - \frac{b(y)}{(1-\gamma(y))^2} \left[\frac{1-\gamma(y)-\lambda y(B(x)-B(y))}{1-\gamma(x)} \right]$$

$$= \frac{df_N}{dy} - \frac{b(y)}{(1-\gamma(y))^2}\, v(y,\,x)$$

changes from negative to positive values with increasing y. By definition, $v(y,\,x)$ equals the expression in square brackets. In analogy to Equation (15) the derivative of $f_N(x)$ can be written in terms of the optimal (N-1)st truncation point, $y_{N-1}^*(x)$,

$$(22) \qquad \frac{df_N(x)}{dx} = \frac{\partial}{\partial x} \frac{\alpha(y_{N-1}^*,\,x)}{(1-\gamma(y_{N-1}^*))(1-\gamma(x))} = \frac{b(x)u_N(x)}{(1-\gamma(x))^2}$$

whence it follows from the defining equation for $v(y\,,\,x)$ that

$$(23a) \qquad u_N(x) = v(x,\, y_{N-1}^*(x)) .$$

By factoring out the term $b(y)\ (1-\gamma(y))^{-2}$ which is common to all terms in Equation (21) we see that y_N^* is either a root of the equation,

$$u_N(y) - v(y,\ x) = 0\ , \tag{23b}$$

or is a point where the left hand side changes discontinuously from negative to positive. To offer a proof, by induction on N, of the monotone convergence of $f_N(x)$, $\frac{df_N}{dx}$ and $y_N^*(x)$ we first show that $v(y, x)$ in Equations (21) and (23) is (i) a convex function of y for fixed values of x, (ii) a non-increasing function of x for fixed $y \geq x$, and (iii) a non-decreasing function of x for fixed $y \leq x$. To show (i) we consider the partial derivatives of $v(y, x)$ with respect to y:

$$\frac{\partial v(y,x)}{\partial y} = -\lambda(1-\gamma(x))^{-1}[B(x) - B(y)] \tag{24a}$$

$$\frac{\partial^2 v}{\partial y^2} = \lambda(1-\gamma(x))^{-1}\, b(y) \geq 0\ . \tag{24b}$$

$v(y, x)$ plotted as a function of y may consist of several straight line segments where $b(y) = 0$; in particular if $b(y) = 0$ in the neighborhood of x, the relative minimum of $v(y, x)$ is not unique. To show (ii) and (iii) we write the partial derivative $v(y, x)$ with respect to x as

$$\frac{\partial v(y,\ x)}{\partial x} = \frac{b(x)}{(1-\gamma(x))^2}\,[\lambda x(1-\gamma(y)) - \lambda y(1-\gamma(x)) - \lambda^2 xy(B(x)-B(y))]. \tag{25a}$$

Integration by parts and cancellation of terms of opposite sign gives

$$\frac{\partial v(y,\ x)}{\partial x} = \frac{\lambda yxb(x)}{(1-\gamma(x)^2}\left[\frac{u_1(y)}{y} - \frac{u_1(x)}{x}\right] \tag{25b}$$

We recall that $u_1(x)/x$ is a decreasing function of x ; hence, $\frac{u_1(y)}{y} < \frac{u_1(x)}{x}$ for $y > x$. The inequality is reversed when $y < x$. The sign of the terms in square brackets in Equation (25b) determines the sign of $\frac{\partial v}{\partial x}$ and, in particular, is only zero when $b(x) = 0$ or $x = y$.

We observe that $u_N(0) = 1$, $u_N(y) > 1$ for $B(y) > 0$. $v(0, x) = (1 - \gamma(x))^{-1} > 1$ and $v(x, x) = 1$.

Hence, there is at least one point $y \leq x$ where $u_N(y) - v(y, x)$ changes from negative to positive values. Denote the points where such a change takes place in increasing order by

$$0 \leq y_N^{(1)}(x) < y_N^{(2)}(x) < \ldots < y_N^{(m)}(x) \leq x \qquad m \geq 1$$

One of these points is $y_N^*(x)$. To show that $y_N^*(x)$ is a non-decreasing function of x , we first note that for $b(t) = 0$ in the interval $x \leq t \leq x + \Delta x$, $g(y, x) = g(y, x + \Delta x)$ for all y , and hence $y_N^*(x + \Delta x) = y_N^*(x)$. For $b(t) > 0$ in the interval, $v(y, x + \Delta x) > v(y, x)$ for all $y \leq x$. Let

$$0 \leq y_N^{(1)}(x + \Delta x) < y_N^{(2)}(x + \Delta x) < \ldots < y_N^{(p)}(x + \Delta x) \leq x$$

be the points where $u_N(y) - v(y, x + \Delta x)$ changes from negative to positive values. Note that

$$y_N^{(1)}(x) \leq y_N^{(1)}(x + \Delta x) \quad \text{and} \quad y_N^{(m)}(x) \leq y_N^{(p)}(x + \Delta x)$$

which establishes $y_N^*(x) \leq y_N^*(x + \Delta x)$ in the case $m = p = 1$. In the general case assume $y_N^*(x + \Delta x) < y_N^*(x)$. Using the properties of $g_N(y, x)$ established in the appendix, we have

$$(26a) \qquad 0 \geq g_N(y_N^*(x)\ ,\ x) - g_N(y_N^*(x + \Delta x)\ x)$$

$$> g_N(y_N^*(x)\ ,\ x + \Delta x) - g_N(y_N^*(x + \Delta x)\ ,\ x + \Delta x) > 0$$

which is a contradiction and therefore implies that $y_N^*(x + \Delta x) \geq y_N^*(x)$.

Since $u_N(x)$ is discontinuous only where $y_N^*(x)$ has discontinuities, the above result implies that discontinuities of $u_N(x)$, at $\bar{x}$, say, are such that $u_N(\bar{x}^-) > u_N(\bar{x}^+)$. Hence, at the points where $u_N(y) - v(y, x)$ changes from negative to positive values, $u_N(y)$ is continuous, and these points are roots of $u_N(y) - v(y, x) = 0$. Two consequences of these results should be noted: $y_N^*(x) < x$; i.e., $y_N^*(x)$ lies in the interior of the interval $0 \leq y \leq x$, and $y_N^*(x) < y_N^*(x + \Delta x)$ unless $b(t) = 0$ in the interval $x \leq t \leq x + \Delta x$.

To prove the monotone convergence of $y_N^*(x)$, we assume that $y_{N-1}^*(x) \leq y_N^*(x)$. Since, for a given $x \geq y$, $v(x, y)$ is a non-increasing function of y

$$(26b) \qquad y_N^*(x) \geq y_{N-1}^*(x) \Longrightarrow u_{N+1}(x) \leq u_N(x)$$

which in turn implies (Equation (22)) that $\frac{df_{N+1}}{dx} \leq \frac{df_N}{dx}$. If there is a unique root to $u_{N+1}(y) - v(y, x) = 0$, then

$$(26c) \qquad u_{N+1}(x) \leq u_N(x) \Longrightarrow y_N^*(x) \leq y_{N+1}^*(x)$$

because the zero crossing of $u_{N+1}(y) - v(y, x)$ has positive slope. In the case of multiple roots label them

$$0 \leq y_N^{(1)}(x) < y_N^{(2)}(x) < \ldots y_N^{(m)}(x) < x$$

$$0 \leq y_{N+1}^{(1)}(x) < y_{N+1}^{(2)}(x) < \ldots y_{N+1}^{(n)}(x) < x$$

Similarly to the single root case (26c)

$$u_{N+1}(x) \leq u_N(x) \Longrightarrow y_N^{(l)}(x) \leq y_{N+1}^{(l)}(x) \quad \text{and} \quad y_N^{(m)}(x) \leq y_{N+1}^{(n)}(x) \; .$$

If we now assume that $y^*_{N+1}(x) < y^*_N(x)$, then from the appendix we also obtain the contradictory inequalities

$$\text{(26d)} \qquad 0 \geq g_N(y^*_N(x),\, x) - g_N(y^*_{N+1}(x),\, x)$$

$$\geq g_{N+1}(y^*_N(x),\, x) - g_{N+1}(y^*_{N+1}(x),\, x) > 0$$

and therefore conclude that $y^*_{N+1}(x) \geq y^*_N(x)$.

Since we have shown (Section 4) that these inequalities (Equations (17), (18)) hold for the $N = 1$ case the proof of the monotone properties of the sequences in Equation (27) is complete:

$$\text{(27a)} \qquad \frac{B(x)}{1-\gamma(x)} = f_1(x) \geq f_2(x) \geq \cdots \geq f_N(x) \geq f_{N+1}(x) \rightarrow f(x) \geqq 0$$

$$\text{(27b)} \qquad \frac{b(x)u_1(x)}{(1-\gamma(x))^2} = \frac{df_1}{dx} \geqq \frac{df_2}{dx} \geqq \cdots \geqq \frac{df_N}{dx} \geqq \frac{df_{N+1}}{dx} \rightarrow \frac{df}{dx} \geqq 0$$

$$\text{(27c)} \qquad 0 \leqq v(x) \leqq y^*_1(x) \leqq \cdots \leqq y^*_{N-1}(x) \leqq y^*_N(x) \rightarrow y^*(x) \leqq x$$

The behavior of the limiting functions $f(x)$ and $y^*(x)$ are discussed in the following section.

6. The $N = \infty$ Case

In the limit of large N we obtain uniform convergence of the optimal policy $y_N^*(x)$ and $f_N(x)$,

$$\lim_{N \to \infty} f_N(x) = f(x) \quad ; \quad \lim_{N \to \infty} y_N^*(x) = y^*(x)$$

Since we have

$$f_{N+1}(x) = g(y_N^*(x), x)$$

$$= f_N(y_N^*(x)) + \frac{\alpha(y_N^*(x), x)}{(1-\gamma(y_N^*(x)))(1-\gamma(x))}$$

for every N and x it can be shown by arguments identical to those presented by Bellman in Chapter IV of Reference (5) that $f(x)$ and $y^*(x)$ are unique solutions of the functional equation.

$$f(x) = \min_{0<y<x} \left\{ f(y) + \frac{\alpha(y, x)}{(1-\gamma(y))(1-\gamma(x))} \right\} \tag{28}$$

A necessary condition for a solution $y^*(x) < x$ is that

$$\frac{\partial g(y, x)}{\partial y} = \lim_{N \to \infty} \frac{\partial g_N(y, x)}{\partial y} = 0 \ ;$$

this statement is equivalent to finding the root of

$$u(y) - v(y, x) = 0 \tag{29}$$

where $u(y) = \lim_{N\to\infty} u_N(y)$ and convergence of $u_N(y)$ follows from Equation (27).

We show that the limiting function $u(x) = 1$ (independent of x) provides a solution of (28). In this case the unique root of (29) is

$y^*(x) = x$ which occurs at the relative minimum of $v(y, x)$ (recall that $v(y, x)$ is convex in y). The limit of both sides of Equation (22) provides us a solution for $f(x)$ by integration; namely,

$$(30) \qquad f(x) = \lim_{N\to\infty} \int_0^x \frac{b(t)u_N(t)}{(1-\gamma(t))^2}\,dt = \int_0^x \frac{dB(t)}{(1-\gamma(t))^2}$$

which also satisfies Equation (28) when $y^*(x) = x$. The interesting feature of this solution for the infinite priority system is that the lowest priority class corresponds to the customers with longest service time, a special case of non-preemptive priority queues studies by Phipps [4]. Stated another way, we find that at each instant in time there are as many priority classes as there are customers in queue. The first customer selected for service is the one with the lowest service time.

7. A Numerical Example

In Figures (2) and (3) we obtain results for a three priority class problem when the a priori distribution function of service times is that given in Figure (1). Service times of customers are equiprobable in the intervals (1, 2) (3, 4) (5, 6) and (7, 8). We observe in Figure 2 that the truncation point, $y_1^*(x)$, separating two priority classes has corners at the end points of these intervals. The results of Section (4) and (5) were used to obtain the corresponding curve for $y_2^*(x)$ in Figure 3. Since $y_2^*(x)$ is the truncation point separating the second and third class the bottom curve in Figure 3 is obtained by substituting $y_2^*(x)$ for x in Figure 2.

Appendix

Assuming a continuous distribution $B(t)$ we can write

$$g_N(y, x) = \int_0^y \frac{\partial g_N(t, x)}{\partial t} + C$$

$$= \int_0^y \frac{b(t)}{(1-\gamma(t))^2} (u_N(t) - v(t, x))dt + f_1(x)$$

To show that the assumption $y_N^*(x + \Delta x) < y_N^*(x)$ leads to (26a) in the case where $v(y, x) < v(y, x + \Delta x)$ let $y_N^{(j-1)}(x + \Delta x) < y_N^*(x) \leq y_N^{(j)}(x + \Delta x)$ and $y_N^{(i)}(x) \leq y_N^*(x + \Delta x) < y_N^{(i+1)}(x)$. It follows from the definition of $y_N^*(x)$ and the fact that $u_N(t) - v(t, x) \geq 0$ in the interval $y_N^{(i)}(x) \leq t \leq y_N(x + \Delta x)$ that

(i) $$0 \geq g_N(y_N^*(x), x) - g_N(y_N^{(i)}(x), x) \geq g_N(y_N^*(x), x) - g_N(y_N^*(x + \Delta x), x)$$

Similarly, from the definition of $y_N^*(x + \Delta x)$ and with $u_N(t) - v(t, x + \Delta x) \leq 0$ in the interval $y_N^*(x) \leq t \leq y_N^{(j)}(x + \Delta x)$ we have

(ii) $$0 < g_N(y_N^{(j)}(x + \Delta x), x + \Delta x) - g_N(y_N^*(x + \Delta x), x + \Delta x) \leq g_N(y_N^*(x), x + \Delta x) - g_N(y_N^*(x + \Delta x), x + \Delta x)$$

Since $u_N(t) - v(t, x) > u_N(t) - v(t, x + \Delta x)$ for $0 \leq t \leq x$ the right hand side of (i) is strictly greater than the right hand side of (ii), which completes (26a).

To prove that the assumption $y_{N+1}^*(x) < y_N^*(x)$ implies (26d), let $y_{N+1}^{(j-1)}(x) < y_N^*(x) \leq y_{N+1}^{(j)}(x)$ and $y_N^{(i)}(x) \leq y_{N+1}^*(x) < y_N^{(i+1)}(x)$.

The same arguments as above establish (26d), replacing $y_N^{(j-1)}(x + \Delta x)$, $y_N^{(j)}(x + \Delta x)$ and $y_N^{*}(x + \Delta x)$ by $y_{N+1}^{(j-1)}(x)$, $y_{N+1}^{(j)}(x)$ and $y_{N+1}^{*}(x)$ respectively, $g_N(y, x + \Delta x)$ by $g_{N+1}(y, x)$, and $u_N(t) - v(t, x + \Delta x)$, by $u_{N+1}(t) - v(t, x)$, and recalling that $u_N(t) - v(t, x) \geq u_{N+1}(t) - v(t, x)$.

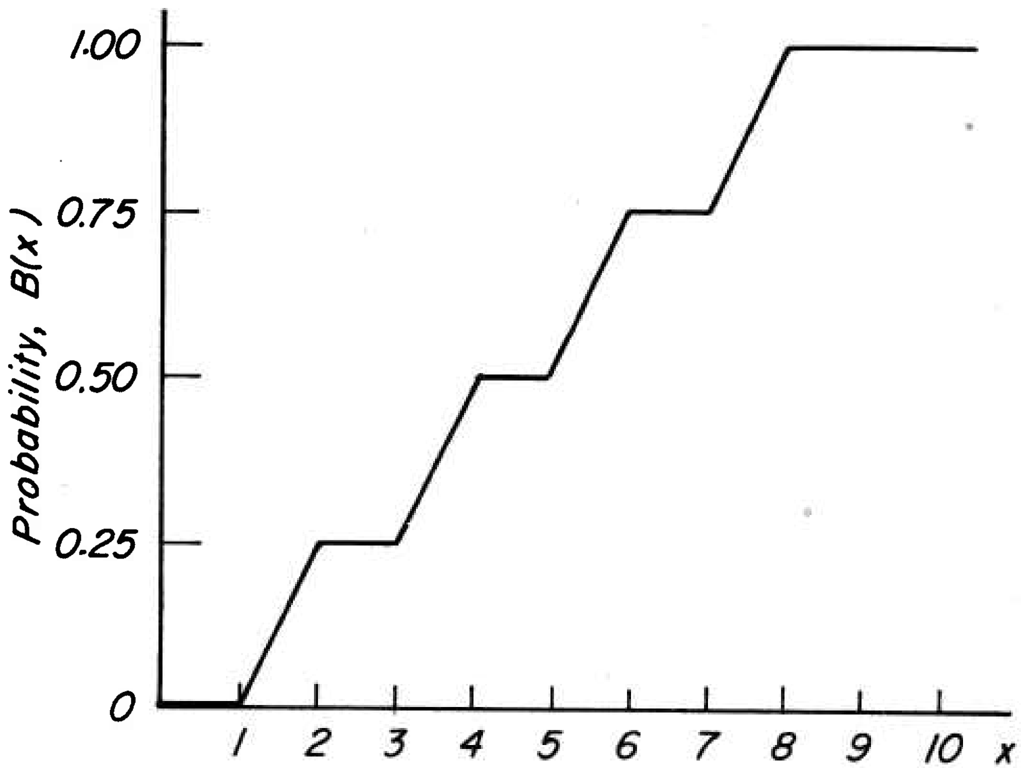

Fig. I – DISTRIBUTION OF SERVICE TIMES

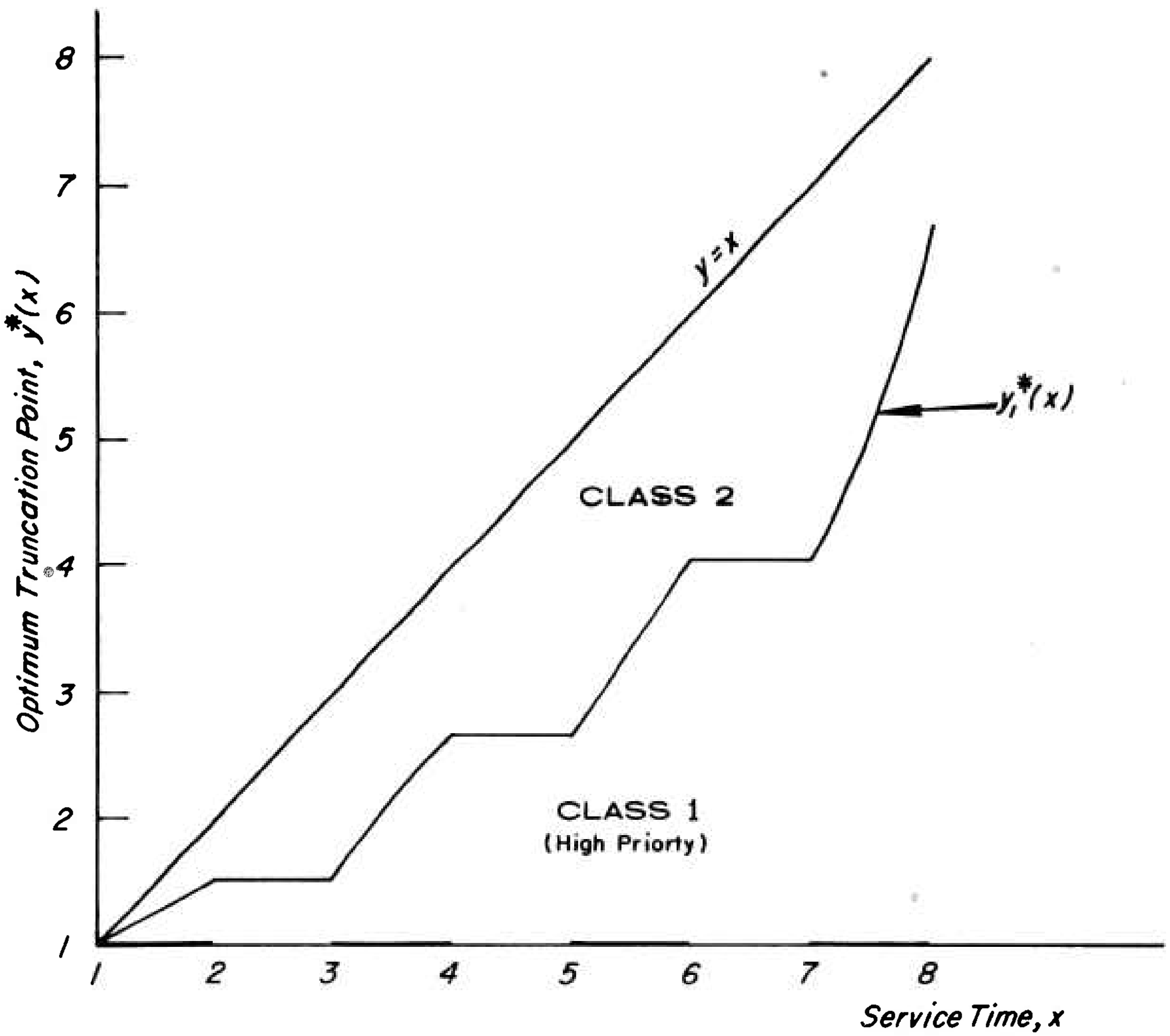

Fig. 2 - LOCATION OF TWO PRIORITY CLASSES.

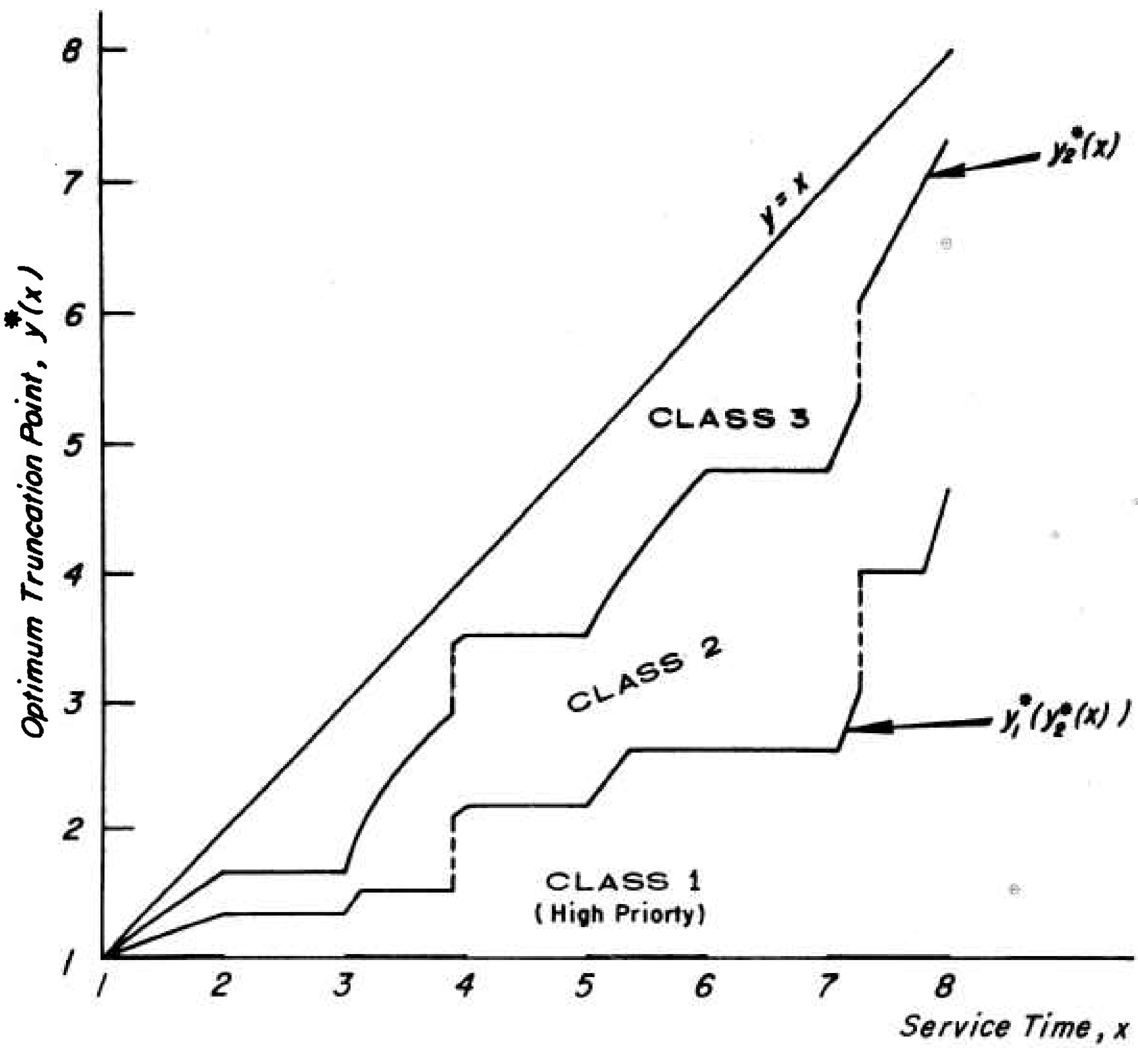

Fig.3 - LOCATION OF THREE PRIORITY CLASSES.

BIBLIOGRAPHY

1. Cobham, A., "Priority Assignment in Waiting Line Problems," Operations Research, Vol. 2 (1954) p. 70-76.

2. Cox, D. R. and W. L. Smith, Queues, John Wiley and Sons (1961).

3. Kesten, H. and J. Runnenberg, "Priority in Waiting Line Problems," Proc. Akad. Wet. Amst. A, Vol. 60 (1957) p. 312-336.

4. Phipps, T. E., "Machine Repair as a Priority Waiting Line Problem," Operations Research, Vol. 4 (1956) p. 76-85.

5. Bellman, R. E., "Dynamic Programming," Princeton University Press (1957).

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